ROOTS OF EHRHART POLYNOMIALS OF GORENSTEIN FANO POLYTOPES

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Abstract. Given arbitrary integers k and d with $0 \leq 2k \leq d$, we construct a Gorenstein Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that (i) its Ehrhart polynomial $i(\mathcal{P}, n)$ possesses d distinct roots; (ii) $i(\mathcal{P}, n)$ possesses exactly $2k$ imaginary roots; (iii) $i(\mathcal{P}, n)$ possesses exactly $d - 2k$ real roots; (iv) the real part of each of the imaginary roots is equal to $-1/2$; (v) all of the real roots belong to the open interval $(-1, 0)$.

Recently, many research papers on convex polytopes, including [2], [3], [4], [5], [7] and [11], discuss roots of Ehrhart polynomials. One of the fascinating topics is the study on roots of Ehrhart polynomials of Gorenstein Fano polytopes.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $\partial\mathcal{P}$ its boundary. (An integral convex polytope is a convex polytope all of whose vertices have integer coordinates.) Given integers $n = 1, 2, \ldots$, we write $i(\mathcal{P}, n)$ for the number of integer points belonging to $n\mathcal{P}$, where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$. In other words,

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad n = 1, 2, \ldots.$$

Late 1950's Ehrhart did succeed in proving that $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$. We call $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of $\mathcal{P}$. Ehrhart's "loi de réciprocité" guarantees that

$$(-1)^d i(\mathcal{P}, -n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|, \qquad n = 1, 2, \ldots.$$

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1}\left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right] = \sum_{i=0}^{\infty} \delta_i \lambda^i.$$

Since $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$, a fundamental fact on generating functions guarantees that $\delta_i = 0$ for every $i > d$. The sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

is called the *δ-vector* of $\mathcal{P}$. Thus $\delta_0 = 1$, $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$ and $\delta_d = |(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Each δ_i is nonnegative (Stanley [14]). If $\delta_d \neq 0$, then $\delta_1 \leq \delta_i$ for every $1 \leq i < d$ ([10]). We refer the reader to [6], [8], [15] [16], [17] and [18] for further informations on Ehrhart polynomials and δ-vectors.

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A *Fano polytope* is an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior $\mathcal{P} \setminus \partial\mathcal{P}$ of $\mathcal{P}$. A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope which consists of those $x \in \mathbb{R}^d$ such that $\langle x, y \rangle \leq 1$ for all $y \in \mathcal{P}$, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$.)

Let $\mathcal{P} \subset \mathbb{R}^d$ be a Fano polytope with $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. It follows from [1] and [9] that the following conditions are equivalent:

- $\mathcal{P}$ is Gorenstein;
- $\delta(\mathcal{P})$ is symmetric, i.e., $\delta_i = \delta_{d-i}$ for every $0 \leq i \leq d$;
- $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $i(\mathcal{P}, n)$ its Ehrhart polynomial. A complex number $a \in \mathbb{C}$ is called a *root* of $i(\mathcal{P}, n)$ if $i(\mathcal{P}, a) = 0$. Let $\Re(a)$ denote the real part of $a \in \mathbb{C}$. An outstanding conjecture given in [2] says that every root $a \in \mathbb{C}$ of $i(\mathcal{P}, n)$ satisfies $-d \leq \Re(a) \leq d-1$.

When $\mathcal{P} \subset \mathbb{R}^d$ is a Gorenstein Fano polytope, since $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$, the roots of $i(\mathcal{P}, n)$ locate symmetrically in the complex plane with respect to the line $\Re(z) = -1/2$. Thus in particular, if d is odd, then $-1/2$ is a root of $i(\mathcal{P}, n)$. It is known [3, Proposition 1.8] that, if all roots $a \in \mathbb{C}$ of $i(\mathcal{P}, n)$ of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d satisfy $\Re(a) = -1/2$, then $\mathcal{P}$ is unimodular isomorphic to a Gorenstein Fano polytope whose (usual) volume is at most 2^d.

Theorem 0.1. *Given arbitrary nonnegative integers k and d with $0 \leq 2k \leq d$, there exists a Gorenstein Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that*

(i) *$i(\mathcal{P}, n)$ possesses d distinct roots;*
(ii) *$i(\mathcal{P}, n)$ possesses exactly $2k$ imaginary roots;*
(iii) *$i(\mathcal{P}, n)$ possesses exactly $d-2k$ real roots;*
(iv) *the real part of each of the imaginary roots is equal to $-1/2$;*
(v) *all of the real roots belong to the open interval $(-1, 0)$.*

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the canonical unit vectors of $\mathbb{R}^d$. Let $\mathcal{Q} \subset \mathbb{R}^d$ be the convex polytope which is the convex hull of $\mathbf{e}_1, \ldots, \mathbf{e}_{2k}$ and $-(\mathbf{e}_1 + \cdots + \mathbf{e}_{2k})$. Then $\mathcal{Q}$ is an integral convex polytope of dimension $2k$ with $\delta(\mathcal{Q}) = (1, 1, \ldots, 1) \in \mathbb{Z}^{2k+1}$. Let $\mathcal{Q}^c \subset \mathbb{R}^d$ be the convex polytope which is the convex hull of $\mathcal{Q} \cup \{\mathbf{e}_{2k+1}, \ldots, \mathbf{e}_d\}$. Then $\delta(\mathcal{Q}^c) = (\delta(\mathcal{Q}), 0, \ldots, 0) \in \mathbb{Z}^{d+1}$. Hence the convex polytope $(d-2k+1)\mathcal{Q}^c$ possesses a unique integer point $\mathbf{a}$ in its interior. Now, write $\mathcal{P} \subset \mathbb{R}^d$ for the integral convex polytope $(d-2k+1)\mathcal{Q}^c - \mathbf{a}$. Then $\mathcal{P}$ is a Gorenstein Fano polytope. Our work is to show that $\mathcal{P}$ enjoys the required properties (i) – (v).

Since

$$\sum_{n=0}^{\infty} i(\mathcal{Q}^c, n)\lambda^n = \frac{1 + \lambda + \lambda^2 + \cdots + \lambda^{2k}}{(1-\lambda)^{d+1}},$$

one has

$$\begin{aligned}
i(\mathcal{Q}^c, n) &= \sum_{i=n-2k}^{n} \binom{d+i}{d} = \sum_{i=0}^{2k} \binom{d+(n-2k)+i}{d} \\
&= \sum_{i=0}^{2k} \binom{d+n-(2k-i)}{d} = \sum_{i=0}^{2k} \binom{n+d-i}{d} \\
&= \sum_{i=0}^{2k} \left(\binom{n+d-i+1}{d+1} - \binom{n+d-i}{d+1} \right) \\
&= \binom{n+d+1}{d+1} - \binom{n+d-2k}{d+1} \\
&= \frac{1}{(d+1)!} (\prod_{i=1}^{d-2k} (n+i)) (\prod_{i=0}^{2k} (n+d+1-i) - \prod_{i=0}^{2k} (n-i)).
\end{aligned}$$

Since

$$i(\mathcal{P}, n) = i((d-2k+1)\mathcal{Q}^c, n) = i(\mathcal{Q}^c, (d-2k+1)n),$$

one has

$$i(\mathcal{P}, n) = \frac{(d-2k+1)^{d+1}}{(d+1)!} (\prod_{i=1}^{d-2k} (n + \frac{i}{d-2k+1})) F(n),$$

where

$$\begin{aligned}
F(n) &= \prod_{i=0}^{2k} (n + \frac{d+1-i}{d-2k+1}) - \prod_{i=0}^{2k} (n - \frac{i}{d-2k+1}) \\
&= \prod_{i=0}^{2k} (n + \frac{d+1-(2k-i)}{d-2k+1}) - \prod_{i=0}^{2k} (n - \frac{i}{d-2k+1}).
\end{aligned}$$

Now, since

$$-\frac{d+1-(2k-i)}{d-2k+1} < -1/2 < \frac{i}{d-2k+1}$$

and since

$$-\frac{d+1-(2k-i)}{d-2k+1} + \frac{i}{d-2k+1} = -1,$$

Lemma 0.2 below guarantees that $F(n)$ possesses $2k$ distinct roots and each of them is an imaginary root with $-1/2$ its real part. Finally, the real roots of $i(\mathcal{P}, n)$ are

$$-\frac{i}{d-2k+1}, \qquad 1 \le i \le d-2k,$$

which belong to the open interval $(-1, 0)$. $\square$

Lemma 0.2. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{2k}$ and $\beta_0, \beta_1, \ldots, \beta_{2k}$ be rational numbers satisfying $\alpha_i < -1/2 < \beta_i$ and $\alpha_i + \beta_i = -1$ for all i. Let*

$$f(x) = \prod_{i=0}^{2k}(x - \alpha_i) - \prod_{i=0}^{2k}(x - \beta_i)$$

be a polynomial in x of degree $2k$. Then $f(x)$ possesses $2k$ distinct roots and each of them is an imaginary root with $-1/2$ its real part.

Proof. We employ a basis technique appearing in [12]. Let $a \in \mathbb{C}$ with $\Re(a) > -1/2$. Since $\alpha_i < -1/2 < \beta_i$ and $(\alpha_i + \beta_i)/2 = -1/2$, it follows that $|a - \alpha_i| > |a - \beta_i|$. Thus $\prod_{i=0}^{2k} |a - \alpha_i| > \prod_{i=0}^{2k} |a - \beta_i|$. Hence $f(a) \neq 0$. Similarly, if $a \in \mathbb{C}$ with $\Re(a) < -1/2$, then $|a - \alpha_i| < |a - \beta_i|$ for all i. Thus $\prod_{i=0}^{2k} |a - \alpha_i| < \prod_{i=0}^{2k} |a - \beta_i|$. Hence $f(a) \neq 0$. Consequently, all roots $a \in \mathbb{C}$ of $f(x)$ satisfy $\Re(a) = -1/2$.

Substituting $y = x + 1/2$ and $\gamma_i = \beta_i + 1/2$ in $f(x)$, it follows that each of the roots $a \in \mathbb{C}$ of the polynomial

$$g(y) = \prod_{i=0}^{2k}(\gamma_i + y) + \prod_{i=0}^{2k}(\gamma_i - y)$$

in y of degree $2k$ satisfied $\Re(a) = 0$. Since $\gamma_i > 0$, one has $g(0) \neq 0$. Hence $g(y)$ possesses no real root. Thus all roots of $f(x)$ are imaginary roots.

What we must prove is that $g(y)$ possesses $2k$ distinct roots. Let $b \in \mathbb{R}$ and $\theta_i(b)$ the argument of $\gamma_i + b\sqrt{-1}$, where $-\pi/2 < \theta_i(b) < \pi/2$. Then $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{\sqrt{-1}\,\theta_i(b)} = -\prod_{i=0}^{2k} e^{-\sqrt{-1}\,\theta_i(b)}.$$

In other words, $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{2\sqrt{-1}\,\theta_i(b)} = -1,$$

which is equivalent to saying that

$$\sum_{i=0}^{2k} \theta_i(b) \equiv \frac{\pi}{2} \pmod{\pi}.$$

Now, we study the function $h(y) = \sum_{i=0}^{2k} \theta_i(y)$ with $y \in \mathbb{R}$. Since $\gamma_i > 0$, it follows that $h(y)$ is strictly increasing with

$$\lim_{y\to\infty} h(y) = k\pi + \pi/2, \quad \lim_{y\to-\infty} h(y) = -(k+1)\pi + \pi/2.$$

Hence the equation

$$h(y) \equiv \frac{\pi}{2} \pmod{\pi}$$

possesses $2k$ distinct real roots, as desired. $\square$

Example 0.3. Let G be a finite connected graph on the vertex set $V(G) = \{1, \ldots, n\}$ with $E(G)$ its edge set. We assume that G possesses no loop and no multiple edge. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the canonical unit vectors of $\mathbb{R}^d$. For an edge $e = \{i, j\}$ of G with $i < j$, we define $\rho(e)$ and $\mu(e)$ of $\mathbb{R}^d$ by setting $\rho(e) = \mathbf{e}_i - \mathbf{e}_j$ and $\mu(e) = \mathbf{e}_j - \mathbf{e}_i$. Write $\mathcal{P}_G \subset \mathbb{R}^d$ for the convex polytope which is the convex hull of $\{\rho(e) : e \in E(G)\} \cup \{\mu(e) : e \in E(G)\}$. Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $\sum_{i=1}^{d} x_i = 0$. Then $\mathcal{P}_G \subset \mathcal{H}$. Identifying $\mathcal{H}$ with $\mathbb{R}^{d-1}$, it turns out that $\mathcal{P} \subset \mathbb{R}^{d-1}$ is a Fano polytope. It then follows from the theory of unimodular matrices (Schrijver [13]) that $\mathcal{P}_G \subset \mathbb{R}^{d-1}$ is a Gorenstein Fano polytope. One of the research problems is to find a combinatorial characterization of the finite graphs G for which all root $a \in \mathbb{C}$ of $i(\mathcal{P}_G, n)$ satisfy $\Re(a) = -1/2$.

For example, if C is a cycle of length 6, then all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_C, n)$ satisfy $\Re(a) = -1/2$. However, if C is a cycle of length 7, then there is a root $a \in \mathbb{C}$ of $i(\mathcal{P}_C, n)$ with $\Re(a) \neq -1/2$.

If G is a tree, then $\mathcal{P}_G$ is unimodular isomorphic to the regular unit crosspolytope which is the convex hull of $\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\}$ in $\mathbb{R}^d$. Hence the δ-vector of $\mathcal{P}_G$ is $\delta(\mathcal{P}_G) = (\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{d})$. Thus by using [12] again all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_G, n)$ satisfy $\Re(a) = -1/2$.

Let G be a complete bipartite graph of type $(2, d-2)$. Thus the edges of G are either $\{1, j\}$ or $\{2, j\}$ with $3 \leq j \leq d$. Let $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$. Then

$$\sum_{k=0}^{d} \delta_k x^k = (1+x)^{d-3}(1 + 2(d-3)x + x^2).$$

It has been conjectured that all roots $a \in \mathbb{C}$ of $i(\mathcal{P}_G, n)$ satisfy $\Re(a) = -1/2$.

References

[1] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), 493–535.

[2] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, *Contemp. Math.* **374** (2005), 15–36.

[3] C. Bey, M. Henk and J. M. Wills, Notes on the roots of Ehrhart polynomials, *Discrete Comput. Geom.* **38** (2007), 81–98.

[4] B. Braun, Norm bounds for Ehrhart polynomial roots, *Discrete Comput. Geom.* **39** (2008), 191–193.

[5] B. Braun and M. Develin, Ehrhart polynomial roots and Stanley's non-negativity theorem, arXiv:0610399.

[6] E. Ehrhart, "Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire," Birkhäuser, Boston/Basel/Stuttgart, 1977.

[7] M. Henk, A. Schürmann and J. M. Wills, Ehrhart polynomials and successive minima, *Mathematika* **52** (2005) 1–16.

[8] T. Hibi, "Algebraic Combinatorics on Convex Polytopes," Carslaw Publications, Glebe, N.S.W., Australia, 1992.

[9] T. Hibi, Dual polytopes of rational convex polytopes, *Combinatorica* **12** (1992), 237–240.

[10] T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, *Adv. in Math.* **105** (1994), 162–165.

[11] J. Pfeifle, Gale duality bounds for roots of polynomials with nonnegative coefficients, arXiv:0707.3010.

[12] F. Rodriguez-Villegas, On the zeros of certain polynomials, *Proc. Amer. Math. Soc.* **130** (2002), 2251–2254.

[13] A. Schrijver, "Theory of Linear and Integer Programming," John Wiley & Sons, 1986.

[14] R. P. Stanley, Decompositions of rational convex polytopes, *Annals of Discrete Math.* **6** (1980), 333–342.

[15] R. P. Stanley, "Enumerative Combinatorics, Volume 1," Wadsworth & Brooks/Cole, Monterey, Calif., 1986.

[16] R. P. Stanley, On the Hilbert function of a graded Cohen–Macaulay domain, *J. Pure and Appl. Algebra* **73** (1991), 307 – 314.

[17] R. P. Stanley, A Monotonicity Property of h-vectors and h^*-vectors, *Europ. J. Combinatorics* **14** (1993), 251–258.

[18] R. P. Stanley, "Combinatorics and Commutative Algebra," Second Ed., Birkhäuser, 1996.

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